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The influence of damping and source terms on solutions of nonlinear wave equations

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ABSTRACT: We discuss in this paper some recent development in the study of nonlinear wave equations. In particular, we focus on those results that deal with wave equations that feature two competing forces. One force is a damping term and the other is a strong source. Our central interest here is to analyze the influence of these forces on the long-time behavior of solutions.

Key Words: degenerate damping, generalized solutions, source terms, wave equations, weak solutions, blow-up of solutions.

Contents

1. Introduction

In this article we describe few recent results associated with nonlinear evolution equations. Our primary interest lies in global existence, finite time blow-up, and regularity of solutions. Such questions have been studied by many authors. In particular, we refer the reader to the monographs [20], [43], the review article [44], and the references therein.

However, until recently, very little work has been done on the problem of global existence, blow-up and asymptotic behavior of solutions to nonlinear wave equations under the influence of nonlinear damping; particularly when the damping term is degenerate. Such evolution equations with damping arise naturally in many contexts. For example, in the context of fluid flows, viscosity effects often appear as damping terms in evolution equations. Also, in classical mechanics, the physical problems of vibrating membranes, strings or shells in elastic media, damping terms reflect the internal energy that is dissipated by the motion.

We consider nonlinear wave equations of the form

$$u_{tt} - \Delta u + \mathcal{G}_0(x,t,u,u_t) = \mathcal{F}_0(x,t,u), \tag{1.1}$$

and systems of the form

$$\begin{aligned} u_{tt} - \Delta u + \mathcal{G}_1(x,t,u,v,u_t) =& \mathcal{F}_1(x,t,u,v), \\ v_{tt} - \Delta v + \mathcal{G}_2(x,t,u,v,v_t) =& \mathcal{F}_2(x,t,u,v), \end{aligned} \tag{1.2}$$

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where the nonlinearities $\mathcal{F}_k$ and $\mathcal{G}_k$, $k = 0, 1, 2$, satisfy certain structural conditions, so that the terms $\mathcal{G}_k$ provide damping and the terms $\mathcal{F}_k$ provide strong sources to the system. Various forms of (1.1)-(1.2) arise in many contexts. If, for example, $\mathcal{G}_0 \equiv 0$ and $\mathcal{F}_0(x, t, u) = u^3$, or more generally, any positive odd power of u, then the equation arises in quantum field theory (see Jörgens [21] and Segal [40]). On the other hand, if $\mathcal{F}_0 \equiv 0$ and $\mathcal{G}_0(x, t, u, u_t) = au_t$, or more generally $au_t + bu_t^3$ then the equation provides *a simple* model for a classical vibrating membrane with a resistance force that is proportional to the velocity u_t. We shall refer to such damping terms in this paper as non-degenerate, i.e., damping terms that depend only on the velocity. Indeed, in this case, equation (1.1) can be treated via the theory of monotone operators and the full well-posedness of strong solutions (in the terminology of monotone operator theory) is now classical [8]. In addition, the presence of $\mathcal{F}_0(x, t, u)$ as a locally Lipschitz source term from $H^1(\Omega)$ into $L_2(\Omega)$ does not affect the arguments for establishing the existence of local solutions via perturbation theory of monotone operators.

However, in general the proportionality coefficients in the damping term $au_t + bu_t^3$ are non-constants and they may depend on the longitudinal displacement u, ∇u and other physical quantities. Studying nonlinear wave equations with a damping term which also depends on the longitudinal displacement $u(x, t)$ is more subtle. Indeed, the dependence on u in the damping term will lead to the degeneracy of the well celebrated monotonicity argument [8,33] which has been the key ingredient for establishing any kind of well-posedness results for many years. For this very reason, we refer to such damping terms as *degenerate damping terms*.

It should be noted here that when damping is absent from the equation, the source term may drive solutions of (1.1) and (1.2) to blow up in finite time. In fact, when $\mathcal{G}_k \equiv 0$, $k = 0, 1, 2$, one can appeal to a variety of methods (see [16,27,35,50]) to show that *most* solutions of (1.1) and (1.2) blow up in finite time. In addition, if the source term is removed from the equation, then damping terms of various forms are known to yield existence of global solutions, (see [2,6,8,19]). However, when both damping and source terms are present in the equation, then the analysis of their interaction and their influence on the global behavior of solutions becomes more difficult. We refer the reader to [9,10,15,28,30,36,38,41,47] and the references therein.

At this end, we remark that throughout the paper, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\Gamma = \partial\Omega$. Also, the following notation will be used in the sequel:

$$|u|_{s,\Omega} \equiv |u|_{H^s(\Omega)} \text{ and } \|u\|_p \equiv \|u\|_{L_p(\Omega)},$$

where $H^s(\Omega)$ and $L_p(\Omega)$ stands for the classical Sobolev spaces and the Lebesgue spaces, respectively.

2. Interior damping and source terms

In 1994 Georgiev and Todorova [15] studied the following initial-boundary value problem:

$$\begin{gathered} u_{tt} - \Delta u + |u_t|^{m-1} u_t = |u|^{p-1} u, \quad \text{in } \Omega \times (0, T) \equiv Q_T, \\ u(x, 0) = u^0(x) \in H_0^1(\Omega), \quad u_t(x, 0) = u^1(x) \in L_2(\Omega), \\ u = 0, \text{ on } \Gamma \times (0, T), \end{gathered} \tag{2.1}$$

where $m, p > 1$. The authors in [15] were able to prove the following important results.

Theorem 2.1. *[15] Let $u^0 \in H_0^1(\Omega), u^1 \in L_2(\Omega)$. Further assume that $p, m > 1$ and $p \leq \frac{n}{n-2}$ if $n > 2$. Then, there exists a unique local weak solution u to (2.1) defined on $[0, T]$ for some $T > 0$,* *with*

$$u \in C([0,T], H_0^1(\Omega)), \;\; u_t \in C([0,T], L_2(\Omega)) \cap L_{m+1}(\Omega \times (0,T)).$$

In addition,

- *If* $p \leq m$, *then the said weak solution is global and* T *may be taken arbitrarily large.*
- *If* $p > m$ *and* $E(0) < 0$, *where* $E(0)$ *is the initial energy given by*

$$E(0) := \frac{1}{2}\left(\|u^1\|_2^2 + \|\nabla u^0\|_2^2\right) - \frac{1}{p+1}\|u^0\|_{p+1}^{p+1},$$

  then, the said weak solution to (2.1) blows up in a finite time.

Although, the statement of the blow up Theorem in [15] required a large negative initial energy, their proof can be adjusted easily and one can obtain the same result without the largeness assumption on $E(0) < 0$. Nonetheless, the blow up result in [15] ignited a lot of interest in wave equations under the influence of both damping and source terms. In fact, the basic calculus of the proof of the blow up Theorem in [15] was later extended by Levine and Serrin [28] to accommodate other abstract evolution equations. In addition, the core ideas of the proof of the blow up Theorem in [15], including the choice of the special Liapunov's function, was used, with the proper adjustment, by many authors. We mention only [3,9,13,36].

In what follows we will discuss few recent results that were inspired by [15]. We begin with some of the results that have appeared in [9].

With $j(s)$ is a continuous, convex real valued function defined on $\mathbb{R}$, j' is its derivative, and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ; the authors in [9] considered the following model:

$$\begin{gathered} u_{tt} - \Delta u + |u|^k j'(u_t) = |u|^{p-1}u, \quad \text{ in } \Omega \times (0,T) \equiv Q_T, \\ u(x,0) = u_0(x) \in H_0^1(\Omega), \quad u_t(x,0) = u_1(x) \in L_2(\Omega), \\ u = 0, \text{ on } \Gamma \times (0,T), \end{gathered} \tag{2.2}$$

where (2.2) is studied under the following conditions imposed on the convex function j and the parameters k, m, p.

Assumption 2.2.

- $k, m, p \geq 0$. *In addition,* $k \leq \frac{n}{n-2}$, $p+1 < \frac{2n}{n-2}$, *if* $n \geq 3$.
- *Coercivity condition:* $j(s) \geq c|s|^{m+1}$, *where* $c > 0$.
- *Strict monotonicity:* $(j'(s) - j'(v))(s-v) \geq c_1|s-v|^{m+1}$, *where* $c_1 > 0$.
- *Continuity:* $|j'(s)| \leq c_0|s|^m + c_2$, *for some constants* $c_0 > 0, c_2 \geq 0$.

A special case of (2.2) is the following well known polynomially damped wave equation studied extensively in the literature (see for instance **[36,38]**).

$$\begin{gathered} u_{tt} - \Delta u + |u|^k |u_t|^{m-1} u_t = |u|^{p-1}u, \quad \text{ in } \Omega \times (0,T) \equiv Q_T, \\ u(x,0) = u_0(x) \in H_0^1(\Omega), \quad u_t(x,0) = u_1(x) \in L_2(\Omega), \\ u = 0, \text{ on } \Gamma \times (0,T), \end{gathered} \tag{2.3}$$

Indeed, by taking $j(s) = \frac{1}{m+1}|s|^{m+1}$, then obviously $j'(s) = |s|^{m-1}s$, and therefore Assumption 2.2 is satisfied. It is also easy to see in this case that problem (2.2) is equivalent to (2.3).

It should be noted here that if $k = 0$ then (2.2) can be treated via perturbation theory of monotone operators [8]. however, the situation is different when the damping term is degenerate ($k > 0$), leading to the degeneracy of the monotonicity argument. In fact, when $k > 0$, (2.3) is no longer a locally Lipschitz perturbation of a monotone problem (even in the case when $p \leq \frac{n}{n-2}$, i.e., the source term is a locally Lipschitz function from $H^1(\Omega)$ into $L_2(\Omega)$). Thus, standard monotone operator theory and the celebrated method of Lions and Strauss [33] do not apply. This fact combined with a potential strong growth of the damping term (the case when $m > 1$), makes the problem interesting mathematically and the analysis more subtle. In fact, one needs to be careful about the meaning of the solution and its relation to the equation.

In discussing finite energy solutions to (2.2) (i.e., $(u, u_t) \in H^1(\Omega) \times L_2(\Omega)$) we need to impose another restriction on the parameters p, m, k:

$$p \leq \max\{\frac{p^*}{2}, \frac{p^*m + k}{m+1}\}; \quad p^* \equiv \frac{2n}{n-2}. \tag{2.4}$$

We should note here that the range of values of the parameter p is beyond what is required for the source term to be a locally Lipschitz function from $H^1(\Omega)$ into $L_2(\Omega)$), as typically assumed in the literature. For instance, in [15] the restriction $p \leq \frac{p^*}{2}$ was crucial, in the non-degenerate case $k = 0$, for establishing the existence of local and global solutions of finite energy on a bounded domain. Instead, condition (2.4) allows "supercritical" values of p provided $p^*(m-1) + 2k > 0$.

The first main result in [9], which establishes local and global existence of generalized solutions, reads as follows:

Theorem 2.3. *[9]* **Generalized solutions.** *Under Assumption 2.2 and condition (2.4), there exists a local* **generalized** *solution to (2.2) defined on $(0,T)$ for some $T > 0$. That is, there exists $T > 0$ and a function u satisfying $u(0) = u_0 \in H_0^1(\Omega)$, $u_t(0) = u_1 \in L_2(\Omega)$, $u \in C_w([0,T], H_0^1(\Omega)) \cap C_w^1([0,T], L_2(\Omega))$ with $|u|^k j(u_t) \in L_1(\Omega \times (0,T))$; and for all $0 < t \leq T$ the following inequality holds:*

$$\begin{aligned} \int_0^t \int_\Omega (u_t\ v_t - \nabla u \nabla v) dx dt + 1/2 \int_\Omega [u_t^2(t) + |\nabla u(t)|^2] dx \\ + \int_0^t \int_\Omega |u|^k [j(u_t) - j(v)] dx dt \\ \leq \int_0^t \int_\Omega |u|^{p-1} u (u_t - v) dx dt + 1/2 \int_\Omega [u_1^2 + |\nabla u_0|^2 - 2u_1 v(0)] dx \end{aligned} \tag{2.5}$$

for all test functions v satisfying

$$v \in H^1(0,T; L_2(\Omega)) \cap L_2(0,T; H_0^1(\Omega)) \cap L_\infty(\Omega \times (0,T)), \quad v(t) = 0.$$

In addition,

- *If $p \leq k + m$, then the said generalized solution is global and T may be taken arbitrarily large.*

- *The said generalized solution satisfies the following energy inequality:*

$$|u(t)|_{1,\Omega} + |u_t(t)|_{0,\Omega} + \int_0^t \int_\Omega |u|^k j(u_t) dx d\tau \le C_T(|u_0|_{1,\Omega}, |u_1|_{0,\Omega}), \tag{2.6}$$

for all $t \in [0,T]$.

We should point out here that the generalized solutions furnished by Theorem 2.3 are obtained without any restriction on parameter m. So, one would wonder how these solutions relate to weak solutions that have been obtained in the literature [36] but only in the special case when $k > 0$, $j(s) = \frac{1}{m+1}|s|^{m+1}$, $m < 1$, and subject to (2.7). It turned out, as stated in Corollary 2.4 below, that for these special cases with additional restrictions imposed on the parameters k, m, the *generalized* solutions in Theorem 2.3 become *unique weak solutions* in the classical sense of weak solutions (i.e., solutions that verify a classical variational *equality*). Thus a particular specialization of Theorem 2.3 to a much narrower range of parameters fully recovers and generalizes (to a larger class of damping functions $j(s)$) the results obtained in [36]. Indeed, it has been shown in [9] a particular specialization of Theorem 2.3 yields the following Corollary.

Corollary 2.4. *[9] In addition to Assumption 2.2 and condition (2.4) further assume that*

$$\begin{cases} m < 1 \text{ if } n = 1, 2; \\ \frac{k}{p^*} + \frac{m}{2} \le \frac{1}{2}, \text{ if } n \ge 3. \end{cases} \tag{2.7}$$

Then, there exists a local weak solution to (2.2) which is defined on an interval $(0,T)$, for some $T > 0$. Moreover,

- *The said solution satisfies the following energy identity:*

$$E(t) := \frac{1}{2}\left(\|u'(t)\|_2^2 + \|\nabla u(t)\|_2^2\right) - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}$$
$$+ \int_0^t \int_\Omega |u(\tau)|^k j'(u'(\tau)) u'(\tau) dx d\tau = E(0). \tag{2.8}$$

- *If $p \le k + m$ then the said solution is global and T may be taken arbitrarily large.*
- *If $k, p \ge 1$ then the said solution is unique and depends continuously on the initial data.*

The authors in [9] also addressed the interesting critical case when $m = 1$. We know that *generalized* solutions exit in this case, as asserted by Theorem 2.3. But do we have *weak* solutions? If so, are these solutions unique? A positive answer to this question is provided by the next theorem.

Theorem 2.5. *[9]* **The case $m = 1$.** *In addition to Assumption 2.2, assume that $k + 1 \le \frac{p^*}{2}$. Then, there exists a local weak solution u to (2.3) (i.e., when $j(s) = \frac{1}{2}|s|^2$ and $j'(s) = s$) such that $u \in C_w([0,T], H_0^1(\Omega)) \cap C_w([0,T], L_2(\Omega))$, $\Delta u - u_{tt} \in L_1(\Omega \times (0,T))$ where u satisfies*

$$\int_0^t \int_\Omega (u_{tt} - \Delta u) v dx dt + \int_0^t \int_\Omega |u|^k j'(u_t) v dx dt$$
$$= \int_0^t \int_\Omega |u|^{p-1} u v dx dt$$
$$u(0) = u_0,\ u_t(0) = u_1 \tag{2.9}$$

for all test functions $v \in L_\infty(\Omega \times (0,T))$, and T may be finite. In addition,

- *If $p \leq k+1$, then the said weak solution is global and T may be taken arbitrarily large.*
- *If $p \geq 1$, then such a solution u is unique, but it may not be continuously dependent on the initial data in the finite energy norm.*

The next issue that was addressed in [9] is the issue of propagation of regularity. This means that more regular data produce more regular solutions. In fact, the result below states that this is always the case locally (i.e., for sufficiently small times). However, in the special case when the parameter p is below the critical value $k+m$, then the propagation of regularity is a global phenomena.

Theorem 2.6. *[9]* **Strong (regular) solutions.** *With the validity of Assumption 2.2, further assume that $n < 5$ and*

$$k \geq 1,\ 2 \leq p < \frac{4}{n-2} + 1,\ m+1 < \frac{n}{n-2},\ k+m < \frac{4}{n-2} + 1.$$

Then, for every initial data satisfying $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, there exists $T_0 > 0$ such that (2.2) has a local solution u with the regularity that $u \in C([0,T], H^2(\Omega)) \cap C^1([0,T], H^1(\Omega))$, for some $T \leq T_0$ where T_0 may be finite.

In addition, if we assume that $p \leq k+m$, $p \leq \frac{p^*}{2}$, and either $k = 0$ or else $\frac{k}{p^*} + \frac{m}{2} \leq \frac{1}{2}$, then the said regular solutions are global and T_0 can be taken arbitrarily large.

Remark 2.7. We note here that the second part of the Theorem 2.6 provides regular solutions in the context of Corollary 2.4. Thus, the weak solutions established in Corollary 2.4 become regular provided the initial data are taken in $H^2(\Omega) \cap H_0^1(\Omega) \times H^1(\Omega)$.

Remark 2.8. For $n < 4$ the proof of Theorem 2.6 allows us to assume that $m+1 \leq \frac{n}{n-2}$ and $k+m \leq \frac{4}{n-2} + 1$ instead of the strict inequalities as assumed in the statement of Theorem 2.6. See [9] for more details.

We now move the discussion towards recent blow up results. The blow up Theorem that appeared in [9,36,38] are more restrictive and they only deal with *weak* solutions that satisfy a variational equality with the added requirement that $\Box u$ and $|u|^k j(u_t) \in L_2(\Omega \times (0,T))$. Indeed, with this added regularity one can establish an energy identity for the weak solutions in [9], which is needed in the proof of the blow-up Theorem. In addition, the mentioned regularity requirement for the weak solutions in [9] forces very strong limitations on the values of m. Since generalized solutions exists for all values of $m > 0$, as asserted by Theorem 2.3, this raises the question of how one obtains a blow up Theorem for generalized solutions.

The first thing we should note here that a direct extension or modification of the proof of the blow-up Theorem given in [9] does not apply for generalized solutions. The reason for this is that the proof in [9] takes advantage of an energy identity (implied by the definition of *weak* solutions), and it also uses the variational form of equation; however *generalized* solutions do not posses, generally, any of these two properties. However, the authors in [11] were able to find necessary conditions (see Theorem 2.9 below) which guarantee that generalized solutions satisfy a suitable variational equality but without the additional requirement that $\Box u \in L_2(\Omega \times (0,T))$ or $|u|^k j(u_t) \in L_2(\Omega \times (0,T))$. Nevertheless, we still do not have an energy identity in [11] as in the case of *weak* solutions.

In order to state Theorem 2.9, we should specify the range of parameters k, p, m for which local generalized solutions do exist, but the condition $k+m \geq p$ (which results in a global solution) is violated. This leads to the following range of parameters:

$$\begin{aligned} k \le \frac{p^*}{2}, \quad p+1 < p^*, \quad p^* = \frac{2n}{n-2} \\ p \le max\{\frac{p^*}{2}, \frac{p^* m + k}{m+1}\}, \quad k+m < p. \end{aligned} \tag{2.10}$$

The main result in [11] reads as follows:

Theorem 2.9. *[11] Let u be generalized solution to (2.2) satisfying (2.5) and assume that the parameters p, m, k satisfy the conditions in (2.10). Further assume that either $u_t \in L_{m+1}(0, T; H_0^1(\Omega))$ or $u_t \in L_s(Q_T)$, where $s \equiv max\{\frac{p^*}{p^*-p}, \frac{p^*(m+1)}{p^*-k}\}$. If $E(0) < 0$, where $E(0)$ is the initial energy given by*

$$E(0) = \frac{1}{2}\left(\|u_1\|_2^2 + \|\nabla u_0\|_2^2\right) - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1};$$

then, the said generalized solution u blows up in a finite time.

Remark 2.10. Any one of the conditions $u_t \in L_{m+1}(0, T; H_0^1(\Omega))$ or $u_t \in L_s(Q_T)$ guarantees that generalized solutions satisfy a suitable variational equality. It should be noted here that neither of the conditions $u_t \in L_{m+1}(0, T; H_0^1(\Omega))$ or $u_t \in L_s(Q_T)$ imply that $|u|^k j(u_t) \in L_2(\Omega \times (0, T))$; nor they imply each other. In addition, the corresponding generalized solutions do not necessarily satisfy an energy identity as in the case of *weak* solutions. Hence, the blow up Theorem in [9] does not cover the class of generalized solutions considered in [11]. We should note also that the class of generalized solutions considered in [11] that satisfies the added regularity assumption that $u_t \in L_{m+1}(0, T; H_0^1(\Omega))$ or $u_t \in L_s(Q_T)$ include the class of *strong* solutions established in [9], at least for some of the range of the parameters.

3. Other related results

Recently, some of the ideas in [9,15,36] have been extended to study certain systems of wave equations. We mention here only the recent results in [3].

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the C^1-function given by

$$F(u, v) = a\,|u+v|^{p+1} + 2b\,|uv|^{\frac{p+1}{2}},$$

where $p \ge 3$, $a > 1$ and $b > 0$. Let $f_1(u, v) = \frac{\partial F}{\partial u}(u, v)$ and $f_2(u, v) = \frac{\partial F}{\partial v}(u, v)$ for $(u, v) \in \mathbb{R}^2$. Throughout the following discussion, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\Gamma = \partial\Omega$; and $n = 1, 2, 3$.

The authors in [3] studied the global well-posedness of the following initial-boundary value problem:

$$\begin{aligned} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u, v), \text{ in } \Omega \times (0, T) \equiv Q_T, \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u, v), \text{ in } \Omega \times (0, T) \equiv Q_T, \\ u(x, 0) = u^0(x) \in H_0^1(\Omega),\ u_t(x, 0) = u^1(x) \in L_2(\Omega), \\ v(x, 0) = v^0(x) \in H_0^1(\Omega),\ v_t(x, 0) = v^1(x) \in L_2(\Omega), \\ u = v = 0, \text{ on } \Gamma \times (0, T). \end{aligned} \tag{3.1}$$

System of nonlinear wave equations such as (3.1) goes back to Reed [39] in 1976 who proposed a similar system in three space dimensions but without the presence of the damping terms $|u_t|^{m-1}u_t$ and $|v_t|^{r-1}v_t$. The nonlinearities $f_1(u,v)$ and $f_2(u,v)$ in (3.1) act as strong source terms in the system (3.1). In addition, the functions f_1, f_2 and F enjoy certain properties. First, it is easy to see that $F(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1})$, for all $(u,v) \in \mathbb{R}^2$, where $c_1 = 2^p a + b$. Moreover, a quick computation will show that for a fixed $a, p > 1$, there exists a constant $c_0 > 0$ such that $F(u,v) \geq c_0(|u|^{p+1} + |v|^{p+1})$, for all $(u,v) \in \mathbb{R}^2$, provided b is chosen large enough. Also, it is easy to see that $uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v)$ for all $(u,v) \in \mathbb{R}^2$. Hence, the following conditions were assumed in [3]:

Assumption 3.1.
- *$m, r \geq 1$; $p \geq 3$ if $n = 1, 2$; $p = 3$ if $n = 3$.*
- *$u^0, v^0 \in H_0^1(\Omega)$, $u^1, v^1 \in L^2(\Omega)$.*
- *There exists constants c_0, $c_1 > 0$ such that*

$$c_0(|u|^{p+1} + |v|^{p+1}) \leq F(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1}) \text{ for all } (u,v) \in \mathbb{R}^2. \tag{3.2}$$

- *In addition,*

$$\begin{aligned} f_1(u,v) &= (p+1)\left[a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}|v|^{\frac{p+1}{2}}u\right], \\ f_2(u,v) &= (p+1)\left[a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}v\right], \\ uf_1(u,v) + vf_2(u,v) &= (p+1)F(u,v) \text{ for all } (u,v) \in \mathbb{R}^2. \end{aligned} \tag{3.3}$$

We can summarize most of the results in [3] in the following Theorem.

Theorem 3.2. *[3] Assume the validity of Assumption 3.1. Then, there exists a unique local weak solution (u,v) to (3.1) defined on $[0,T]$ for some $T > 0$; with $u, v \in C([0,T], H_0^1(\Omega))$, $u_t, v_t \in C([0,T], L^2(\Omega))$, $u_t \in L^{m+1}(\Omega \times (0,T))$, $v_t \in L^{r+1}(\Omega \times (0,T))$, and u satisfies the following energy identity:*

$$E(t) + \int_0^t \int_\Omega |u'(\tau)|^{m+1} dx d\tau + \int_0^t \int_\Omega |v'(\tau)|^{r+1} dx d\tau = E(0), \tag{3.4}$$

where

$$\begin{aligned} E(t) := &\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \\ &- \int_\Omega F(u(t), v(t)) dx. \end{aligned} \tag{3.5}$$

In addition

- *If $p \leq \min\{m, r\}$, then the said weak solution (u,v) is global and T may be taken arbitrarily large.*
- *If $p > \max\{m, r\}$ and $E(0) < 0$, where $E(0)$ is the initial energy given by*

$$E(0) := \frac{1}{2}\left(\|u^1\|_2^2 + \|v^1\|_2^2 + \|\nabla u^0\|_2^2 + \|\nabla v^0\|_2^2\right) - \int_\Omega F(u^0, v^0) dx;$$

then, the said weak solution (u,v) blows up in finite time.

More recently, Alves and Cavalcanti [4] studied the global existence, uniform stabilization and blow up in finite time, for solutions of the following damped nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + h(u_t) = g(u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega, \end{cases} \tag{3.6}$$

where Ω is a bounded domain of $\mathbb{R}^2$ having a smooth boundary $\partial\Omega = \Gamma$, g is (a source) assumed to have a exponential growth at the infinity and h is a monotonic continuous increasing function.

As it was emphasized in [4], there is a strong connection between the well-posedness of (3.6) and the theory of elliptic problems, particularly the well known *mountain passel level.* More precisely, we define the functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(u)\,dx, \tag{3.7}$$

where $G(u) = \int_0^u g(s)\,ds$. The critical points of the functional J are the weak solutions of the elliptic problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \Gamma. \end{cases}$$

Also, related to the functional J is the well known *Nehari manifold* given by:

$$\begin{aligned} \mathcal{N} &:= \big\{u \in H_0^1(\Omega) :\ J'(u)u = 0,\ u \neq 0\big\} \\ &= \{u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2\,dx = \int_\Omega g(u)\,u\,dx,\ u \neq 0\}. \end{aligned} \tag{3.8}$$

If g satisfies suitable properties, it is possible to prove that the mountain pass level d satisfies the following equality (for instance, see Willem [53])

$$d := \inf_{u\in\mathcal{N}} J(u). \tag{3.9}$$

Put

$$W = \{u \in H_0^1(\Omega); J(u) < d\}, \tag{3.10}$$

$$W_1 = \{u \in W : \int_\Omega |\nabla u|^2\,dx > \int_\Omega g(u)u\,dx\} \cup \{0\}, \tag{3.11}$$

$$W_2 = \{u \in W : \int_\Omega |\nabla u|^2\,dx < \int_\Omega g(u)u\,dx\}. \tag{3.12}$$

The energy associated to (3.6) is given by

$$E(t) := \frac{1}{2}\int_\Omega |u_t(x,t)|^2\,dx + J(u(t)). \tag{3.13}$$

By assuming $E(0) < d$, the authors in [4] were able to prove the existence of a global weak solution to (3.6) if the initial datum $u^0 \in W_1$, $u^1 \in L^2(\Omega)$. On the other hand, if $u^0 \in W_2$ then local weak solutions to (3.6) blow up in finite time. We should mention here that the main ingredients used in the proofs in [4] are the Trudinger-Moser inequality (see [34,49]) and the well known Mountain Pass level due to Ambrosetti and Rabinowitz [5].

In order to state some of the results established in [4] we list the technical conditions on the functions g and h.

Assumption 3.3. *[4]*

Assume that $g : \mathbb{R} \to \mathbb{R}$ is a C^1 function and $h : \mathbb{R} \to \mathbb{R}$ is a monotonic increasing continuous function verifying:

- *For each $\beta > 0$, there exists a positive constant C_β such that*

$$|g'(t)|, |g(t)| \le C_\beta e^{\beta t^2}, \quad \text{for all} \;\; t \in \mathbb{R}. \tag{3.14}$$

- *In addition, the function g satisfies the following condition near the origin*

$$\lim_{t\to 0} \frac{g(t)}{t} = 0. \tag{3.15}$$

- *The function $g(t)/t$ is a increasing function in $(0, +\infty)$.*
- *There exist constants C_0, $C_1 > 0$ such that the function h satisfies:*

$$h(t)\, t \ge 0 \quad \text{for all } t \in \mathbb{R}, \tag{3.16}$$

$$h(t)t \le C_0|t|^q, \quad \text{for all } |t| \ge 1 \;\; \text{and for some} \;\; q > 1, \tag{3.17}$$

$$(h(s) - h(t))(s - t) \ge C_1 \left(|s|^{q-2}\, s - |t|^{q-2}\, t\right)(s - t), \;\text{for all } s, t \in \mathbb{R}. \tag{3.18}$$

An example of such a function g is

$$g(t) := |t|^{p-2} t\, e^{C\,|t|^\alpha} \quad \text{for all } t \in \mathbb{R},$$

where $p > 2$, $C > 0$ and $\alpha \in (1, 2)$ is fixed.

The first main result in [4] which establishes the existence of local solutions of (3.6) reads as follows.

Theorem 3.4. *[4] Assume the validity of Assumption 3.3. Then, there exists a unique weak solution u of (3.6) defined on $[0, T]$, for some $T > 0$ such that*

$$u \in C^0([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)), \;\; u_t \in L^q(\Omega \times (0, T)).$$

In addition, the said weak solution u satisfies the energy identity:

$$E(t) - E(s) + \int_s^t \int_\Omega h(u_t) u_t \, dx\, d\tau = \int_s^t \int_\Omega g(u)\, u_t \, dx\, d\tau,$$

for $0 \le s \le t \le T$.

In order to obtain a global weak solution to (3.6), it is necessary to enforce one more condition on the function g.

Assumption 3.5. *[4] There exists $\theta > 2$ such that*

$$0 < \theta G(t) < g(t)\, t \quad \text{for all} \quad t \in \mathbb{R} \setminus \{0\}. \tag{3.19}$$

Remark 3.6. We note here that (3.19) is the well known Ambrosetti-Rabinowitz condition, widely used in elliptic problems. In addition, the assumptions (3.14), (3.15) and (3.19) guarantee that the level d of the mountain pass can be characterized by expression given by (3.9).

The next result in [4] reads as follows.

Theorem 3.7. *[4] In addition to the validity of Assumption 3.3 and 3.5, assume that $(u^0, u^1) \in W_1 \times L^2(\Omega)$ and $E(0) < d$. Then, the weak solution u of (3.6) furnished by theorem 3.4 is a global solution and T may be taken arbitrarily large. Moreover, $u(t) \in W_1$ for all $t \geq 0$.*

Before stating the stability result in [4], we need to introduce certain important functions. Such functions were first introduced in the literature by Lasiecka and Tataru [26] for attractive forces and recently extended by Cavalcanti, Domingos Cavalcanti and Lasiecka [13] for repulsive forces (sources). Following [13,26], Let h^* be a concave, strictly increasing function, with $h^*(0) = 0$, and such that

$$h^*\left(s\, h(s)\right)) \geq s^2 + h^2(s), \text{ for } |s| < 1. \tag{3.20}$$

We note here that with the hypotheses on the h in Assumption 3.3, the construction of such a function h^* is straightforward. With this function, define

$$r(.) = h^*\left(\frac{.}{meas\,(Q_T)}\right), \; Q_T = \Omega \times (0, T). \tag{3.21}$$

As r is strictly increasing, then $cI + r$ is invertible for all $c \geq 0$. For K a positive constant, put

$$p(x) = (cI + r)^{-1}(Kx)\,, \; K := (C_{\theta,d}\, meas(Q_T))^{-1}, \tag{3.22}$$

where $C_{\theta,d}$ is a positive computable constant (see [4] for more details).

Clearly the function p is positive, continuous and strictly increasing with $p(0) = 0$. Finally, put

$$q(x) = x - (I + p)^{-1}(x)\,. \tag{3.23}$$

Now, we are able to state the following stability result in [4].

Theorem 3.8. *[4] Assume the validity of the assumptions of Theorem 3.7 with $q = 2$ (see (3.17)). Let u be the global weak solution of problem (3.6) furnished by Theorem 3.7. With the energy $E(t)$ as defined in (2.8), then there exists $T_0 > 0$ such that*

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0, \tag{3.24}$$

with $\lim_{t\to\infty} S(t) = 0$, where the contraction semigroup $S(t)$ is the solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0), \tag{3.25}$$

where q is as given in (3.23). Here the constant c (from definition (3.22)) is taken to be $c \equiv \frac{1}{meas(Q_T)}$.

In order to state the blow up result in [4], the following additional Assumption will be needed.

Assumption 3.9. *[4]*

- *Suppose that*
$$E(0) \leq 0 \text{ or } 0 \leq E(0) < \gamma\, d, \qquad \text{for some } \gamma \in (0,1) \tag{3.26}$$
with $\gamma \approx 0$ *and* d *is the level of the mountain pass associated with the corresponding elliptic problem.*
- *Assume*
$$h(t) = t. \tag{3.27}$$
- *There exists* $\theta > 3 + \frac{1}{\lambda_1}$ *such that*
$$0 < \theta\, G(t) < g(t)\, t \qquad \text{for all } t \in \mathbb{R} \setminus \{0\}, \tag{3.28}$$

where λ_1 *represents the first eigenvalue of* $-\Delta$ *operator with a Dirichlét boundary condition.*

Theorem 3.10. *[4] With the validity of Assumption 3.3 and Assumption 3.9, further assume that* $(u^0, u^1) \in W_2 \times L^2(\Omega)$*. Then, there exists* $T > 0$ *such that* u *blows-up in the* L^2 *norm of* Ω *as* $t \to T^-$.

Remark 3.11. The linear damping $h(t) = t$ in Theorem 3.10 is not strong enough to overcome the very strong source in (3.6). Indeed, if $E(0) < 0$, then one can modify the proof in [15] to obtain a blow up result, without having the assumption that $u^0 \in W_2$.

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